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On nonuniform Euler–Bernoulli and Timoshenko beams with jump discontinuities: application of distribution theory

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Abstract

In this article, bending of nonuniform Euler–Bernoulli and Timoshenko beams with jump discontinuities in the slope, deflection and mechanical properties are studied. The governing equations are obtained in the space of generalized functions, and the expression of its governing differential equations in terms of a single displacement function and a single rotation function is shown always to be possible. In contrast, for a nonuniform Euler–Bernoulli beam with jump discontinuities in slope and deflection and abrupt changes in flexural stiffness, the governing equation can be written in terms of a single displacement function only under certain conditions. It is observed that for most discontinuous nonuniform Euler–Bernoulli beams we cannot write the governing differential equation in terms of a single displacement function: usually, if there are n discontinuity points on a nonuniform Euler–Bernoulli beam, $n+1$ displacement functions appear in the governing equilibrium equation.

Keywords: Generalized functions; Distribution theory; Nonuniform beams

1. Introduction

In an optimal design, structural sections are usually nonuniform. Hence, the analysis of nonuniform structural forms, such as beams, is of practical importance. Analytical solutions for nonuniform beams have been obtained by several authors (Hetényi, 1937; Fertis and Keene, 1990; Lee et al., 1990; Romano and Zingone, 1992; Al-Gahtani and Khan, 1998). In all these investigations continuous, nonuniform beams were considered. Obviously, when a nonuniform beam has jump discontinuities in deflection or slope and/or has abrupt changes in mechanical properties, the analysis techniques that are used for continuous non-uniform beams should be modified.

There are two alternatives for analyzing discontinuous beams. The first approach, which is commonly used, is to partition the beam into continuous beam segments. In this method the analysis has two steps: in

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Nomenclature

$EI(x)$ flexural stiffness of a nonuniform Euler–Bernoulli or Timoshenko beam
$EI_1(x)$, $EI_2(x)$ flexural stiffness of nonuniform beam segments
$GA_s(x)$ shear stiffness of a nonuniform Timoshenko beam
$GA_{s1}(x)$, $GA_{s2}(x)$ shear stiffness of nonuniform Timoshenko beam segments
$J_i(x)$, $K_i(x)$ functions defined for nonuniform Euler–Bernoulli beams
K_{r0} stiffness of a rotational spring
K_s shear correction factor for a Timoshenko beam
K_{t0} stiffness of a translational spring
L length of a beam
M, M_1, M_2 bending moments
N a positive integer
P_1, P_2 axial forces
V_1, V_2 shear forces
a_0, b_0 constants for nonuniform beams
${}^ib_k(x)$ function $b_k(x)$ for the ith subsystem of a mechanical system
$q(x)$ a distributed force
u_1, u_2, u_3 displacement components
x longitudinal axis of a beam
x_0 position of a discontinuity point
x_0^+ $x_0 + \varepsilon$ for a very small positive ε
x_0^- $x_0 - \varepsilon$ for a very small positive ε
$w^E(x)$ deflection of a nonuniform Euler–Bernoulli beam
$w^T(x)$ deflection of a nonuniform Timoshenko beam
Δ_0 strength of a jump discontinuity in deflection of a nonuniform Euler–Bernoulli beam
Δ_0^T strength of a jump discontinuity in deflection of a nonuniform Timoshenko beam
$\Delta_0^{(i)}$ strength of a jump discontinuity in the ith derivative of a function $f(x)$ at $x = x_0$
Θ_0 strength of a jump discontinuity in the slope of a nonuniform Euler–Bernoulli beam
Θ_0^T strength of a jump discontinuity in the slope of a nonuniform Timoshenko beam
$\Phi_1(x)$, $\Phi_2(x)$ slope of nonuniform Timoshenko beam segments
$\alpha_i(x)$, $\beta_i(x)$, $\gamma_i(x)$ functions defined for nonuniform Timoshenko beams
H a mechanical system

Mathematical symbols
D set of all good functions
$H(x - x_0)$ Heaviside's unit step function
$\Im$, $\Re$ abbreviations for left side of some equations
| set of positive integers
O Launda's order symbol
f, g distributions
$f(x)$ a classic function
$f^G(x)$ the generalized function corresponding to $f(x)$
$\delta(x - x_0)$ Dirac delta function
$\delta^{(n)}(x - x_0)$ the nth distributional derivative of the Dirac delta function
φ a good function
ψ a fairly good function

the first step each beam segment is analyzed separately, and in the second step continuity and boundary conditions are enforced.

The second approach, which was recently introduced by Yavari et al. (2000, 2001), and Yavari and Sarkani (2001), is to formulate the boundary value problem describing the beam bending in the space of generalized functions. Because in the space of generalized functions piecewise, continuous functions have derivatives, the bending of beams with jump discontinuities can be formulated without the need of partitioning the beam into continuous beam segments.

In this article, the bending of nonuniform Euler–Bernoulli and Timoshenko beams with jump discontinuities is studied in the space of generalized functions. We show that the governing equilibrium equations of a nonuniform Timoshenko beam can always be expressed in terms of a single deflection and a single rotation function, and we obtain the explicit forms of the governing differential equilibrium equations. Then the bending of nonuniform Euler–Bernoulli beams with jump discontinuities is studied in the space of generalized functions. Surprisingly, unlike the Timoshenko beam's governing equilibrium equation, that of an Euler–Bernoulli beam with jump discontinuities cannot always be written in terms of a single deflection function. Conditions under which the governing equilibrium equation of the beam can be simplified to a differential equation in terms of a single deflection function are obtained. The mathematical reason is shown to be the appearance of higher-order derivatives in the governing equilibrium equation of the Euler–Bernoulli beam, and the lack of a definition for the product of generalized functions.

This paper is structured as follows. Section 2 presents some definitions and concepts of the theory of generalized functions used in our investigation. In Section 3, the governing equilibrium equations of a nonuniform Timoshenko beam with jump discontinuities are obtained and compared with those of continuous Timoshenko beams. The same problem is studied for Euler–Bernoulli beams in Section 4. Conclusions are given in Section 5.

2. Theory of generalized functions

In classical mathematical analysis, continuous functions are not necessarily differentiable. Distributions (or generalized functions), roughly speaking, generalize the concept of functions such that a continuous generalized function becomes differentiable; each generalized function is differentiable and its derivative is another generalized function. There are two methods for defining generalized functions. The first method is the method of functionals, used by Sobolev (1938) and Schwarz (1966). A disadvantage of this method is its complexity, which can be daunting for users who are not mathematicians. The second method, which has been adopted more recently, is to consider generalized functions as limits of a sequence of functions. The sequential approach is not only simpler, but also more in agreement with the intuition of physicists. Here we use the sequential method for defining generalized functions, following Lighthill (1958) and Jones (1982). For an alternative method of definition, the reader may refer to Zemanian (1965).

Definition 1. A good function is infinitely smooth and the function and all its derivatives are $\mathrm{O}(|x|^{-N})$ as $|x| \to \infty$ for all $N \in |$. The set of all good functions is denoted by D.

As an example, $f(x) = \mathrm{e}^{-x^2}$ is a good function.

Definition 2. A sequence $\{f_n\}$ of good functions is said to be regular if, for every good function φ, the limit

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} f_n(x)\varphi(x)\,\mathrm{d}x \tag{1}$$

exists and is finite.

Definition 3. Two regular sequences of good functions $\{f_n\}$ and $\{g_n\}$ are equivalent if for every good function φ:

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty} f_n(x)\varphi(x)\,\mathrm{d}x = \lim_{n\to\infty}\int_{-\infty}^{+\infty} g_n(x)\varphi(x)\,\mathrm{d}x \tag{2}$$

Definition 4. An equivalent class of regular functions is a generalized function.

Definition 5. If two generalized functions f and g are defined by sequences $\{f_n\}$ and $\{g_n\}$, $f+g$ is a generalized function defined by the sequence $\{f_n+g_n\}$.

Definition 6. A fairly good function is infinitely smooth and the function and all its derivatives are $\mathrm{O}(|x|^N)$ as $|x|\to\infty$ for some $N \in |$.

Polynomials are examples of fairly good functions.

Definition 7. The product of a generalized function f and a fairly good function ψ is defined by the sequence $\{f_n\psi\}$.

Definition 8. If $f(x)$ is a function of x in the ordinary sense and $(1+x^2)^{-N}f(x)$ is absolutely integrable in $(-\infty,+\infty)$ for some $N \in |$, then the generalized function $f^{\mathrm{G}}(x)$ is defined by a sequence $\{f_n\}$ such that:

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty} f_n(x)\varphi(x)\,\mathrm{d}x = \lim_{n\to\infty}\int_{-\infty}^{+\infty} f(x)\varphi(x)\,\mathrm{d}x \quad \forall\varphi\in D \tag{3}$$

Definition 9. The sequence $\{(n/\pi)^{1/2}\mathrm{e}^{-nx^2}\}$ is regular and defines a generalized function $\delta(x)$ such that:

$$\int_{-\infty}^{+\infty} \delta(x)\varphi(x)\,\mathrm{d}x = \varphi(0) \quad \forall\varphi\in D \tag{4}$$

This generalized function is known as the Dirac delta function (Dirac, 1930). The sequences defining δ, $\delta^{(1)}$, $\delta^{(2)}$, and $\delta^{(3)}$ are shown in Fig. 1.

Similarly, $\delta(x-x_0)$ is defined as:

$$\int_{-\infty}^{+\infty} \delta(x-x_0)\varphi(x)\,\mathrm{d}x = \varphi(x_0) \quad \forall\varphi\in D \tag{5}$$

Definition 10. A generalized function $f(x)$ is said to be even (or odd, respectively) if

$$\int_{-\infty}^{+\infty} f(x)\varphi(x)\,\mathrm{d}x = 0 \tag{6}$$

for all odd (or even, respectively) good functions in D.

The Dirac delta function is an even generalized function.

Suppose that $f(x)$ is a generalized function and $f(x_0)$ is defined. The product of $f(x)$ and $\delta(x-x_0)$ is defined as:

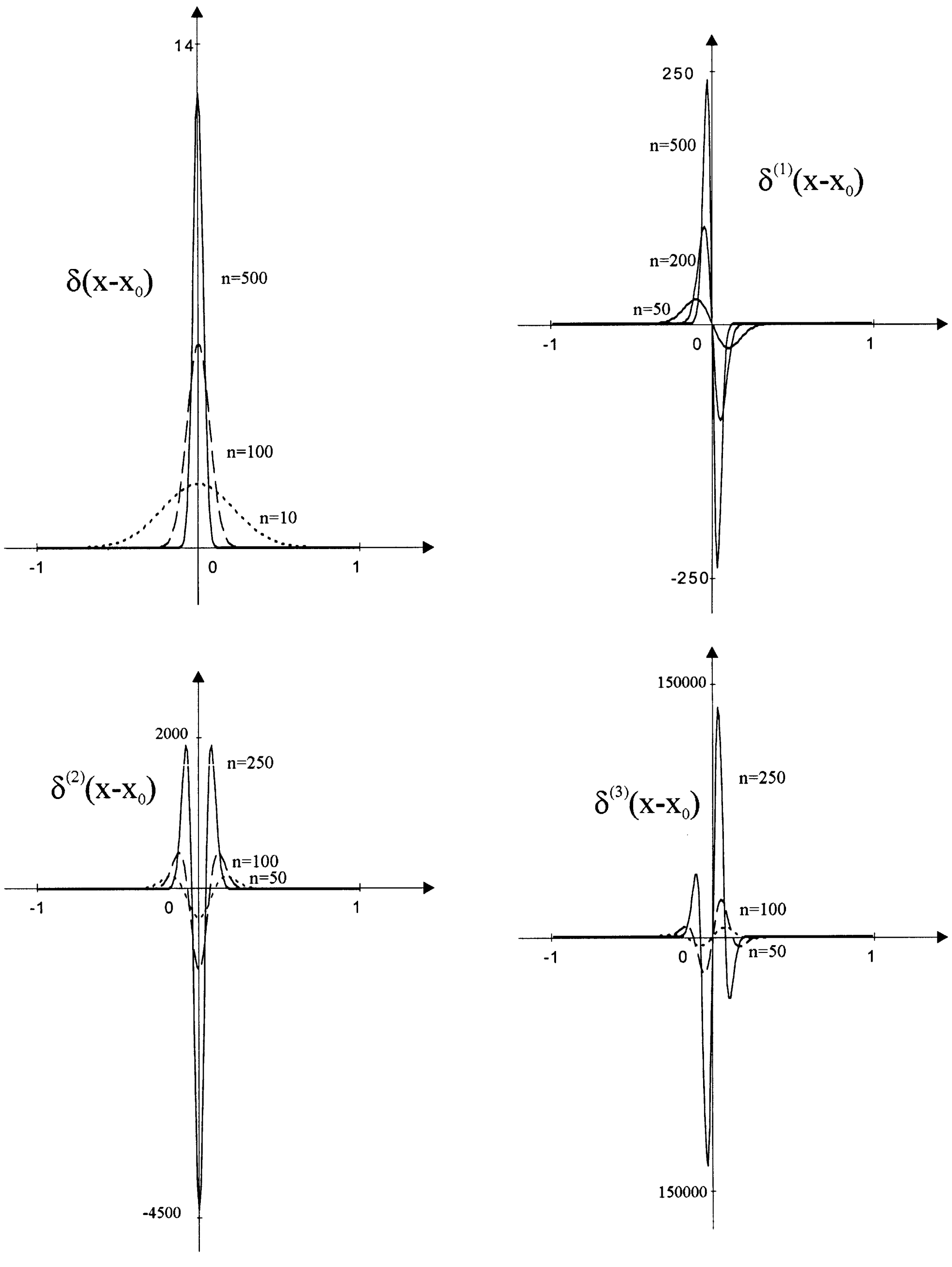

Fig. 1. Functions in the sequence $\sqrt{n/\pi}e^{-nx^2}$ used to define Dirac delta function and its first three derivatives.

$$\int_{-\infty}^{+\infty} (\delta(x-x_0)f(x))\varphi(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \delta(x-x_0)(f(x)\varphi(x))\,\mathrm{d}x = f(x_0)\varphi(x_0)$$
$$= \int (f(x_0)\delta(x-x_0))\varphi(x)\,\mathrm{d}x \tag{7}$$

Hence:

$$f(x)\delta(x-x_0) = f(x_0)\delta(x-x_0) \tag{8}$$

Definition 11. The sequence $\{\mathrm{d}f_n(x)/\mathrm{d}x\}$ is regular and defines a generalized function denoted by $f^{(1)}(x)$ and called the distributional derivative of *f*.

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}f_n(x)}{\mathrm{d}x}\varphi(x)\,\mathrm{d}x = f_n(x)\varphi(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f_n(x)\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\,\mathrm{d}x \tag{9}$$

Because $\varphi(x)f_n(x)$ is a good function, it vanishes at $\pm\infty$. Also $\mathrm{d}\varphi(x)/\mathrm{d}x$ is a good function. Hence:

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}f_n(x)}{\mathrm{d}x}\varphi(x)\,\mathrm{d}x = -\int_{-\infty}^{+\infty} f_n(x)\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\,\mathrm{d}x \quad \forall\varphi\in D \tag{10}$$

Similarly:

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}^n f_n(x)}{\mathrm{d}x}\varphi(x)\,\mathrm{d}x = (-1)^n\int_{-\infty}^{+\infty} f_n(x)\frac{\mathrm{d}^n\varphi(x)}{\mathrm{d}x}\,\mathrm{d}x \quad \forall\varphi\in D \tag{11}$$

Corollary 1. *For the delta function we have:*

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}^n\delta(x-x_0)}{\mathrm{d}x}\varphi(x)\,\mathrm{d}x = (-1)^n\varphi^{(n)}(x_0) \quad \forall\varphi\in D \tag{12}$$

Definition 12. The Heaviside function $H(x-x_0)$ is defined as:

$$H(x-x_0) = \begin{cases} 0; & x<x_0 \\ 1; & x>x_0 \end{cases} \tag{13}$$

This function has a jump discontinuity at $x=x_0$. Its value at this point is usually taken to be one-half. The reason for choosing one-half at the point of discontinuity is explained later. The Heaviside function is very useful in working with functions with jump discontinuities. As an example, consider a function $f(x)$ which is continuous everywhere on the real line except at the point $x=x_0$, where it has a jump discontinuity, i.e.:

$$f(x) = \begin{cases} f_1(x); & x<x_0 \\ f_2(x); & x>x_0 \end{cases} \tag{14}$$

This function can be written in a more compact form as:

$$f(x) = f_1(x) + [f_2(x) - f_1(x)]H(x-x_0) \tag{15}$$

Substituting $H(x-x_0)$ in Eq. (10) yields:

$$\int_{-\infty}^{+\infty} \frac{\bar{\mathrm{d}}H(x-x_0)}{\mathrm{d}x}\varphi(x)\,\mathrm{d}x = -\int_{-\infty}^{+\infty} H(x-x_0)\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\,\mathrm{d}x = -\int_{x_0}^{+\infty}\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\,\mathrm{d}x = \varphi(x_0)$$
$$= \int_{-\infty}^{+\infty}\delta(x-x_0)\varphi(x)\,\mathrm{d}x \tag{16}$$

Thus:

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}x}H(x-x_0) = \delta(x-x_0) \tag{17}$$

where a bar over the distributional differentiation symbol distinguishes it from the classical differentiation. Now we show why the value of Heaviside's function at the discontinuity point is assumed to be one-half. It is known that the Dirac delta function is an even function. Hence:

$$\int_{x_0}^{+\infty} \delta(x - x_0)\,\mathrm{d}x = \frac{1}{2} = \int_{-\infty}^{+\infty} \delta(x - x_0) H(x - x_0)\,\mathrm{d}x = H(x - x_0)|_{x=x_0} \tag{18}$$

It is desirable to define the product of two generalized functions f_1 and f_2 in a manner consistent with a definition of the conventional product $f_1 f_2$ when f_1 and f_2 are classical functions. Unfortunately, it is not possible to define such a product for all generalized functions. However, it is possible to define the product of generalized function f and a fairly good function ψ. The problem with generalized functions is that they are not defined pointwise. In some cases the product of two generalized functions is possible. As an example, consider the following two products:

$$\delta(x - x_1)\delta(x - x_2) = \begin{cases} 0; & x_1 \neq x_2 \\ \text{not defined}; & x_1 = x_2 \end{cases} \tag{19a}$$

and

$$\delta(x - x_1)H(x - x_2) = \begin{cases} H(x_1 - x_2); & x_1 \neq x_2 \\ \frac{1}{2}\delta(x - x_1); & x_1 = x_2 \end{cases} \tag{19b}$$

Definition 13. If f_1 and f_2 are generalized functions and $\{f_{n1}\}$ and $\{f_{n2}\}$ are sequences defining them, and if $\{f_{n1} f_{n2}\}$ is a regular sequence, $f_1 f_2$ is defined by the sequence $\{f_{n1} f_{n2}\}$, and:

$$\int_{-\infty}^{+\infty} (f_{n1} f_{n2})(x)\varphi(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} f_{n1}(x) f_{n2}(x)\varphi(x)\,\mathrm{d}x \quad \forall \varphi \in D \tag{20}$$

Theorem 1. *If f is a generalized function and if $f(x_0)$ is defined, then:*

$$f(x)\delta^{(1)}(x - x_0) = f(x_0)\delta^{(1)}(x - x_0) - f^{(1)}(x_0)\delta(x - x_0) \tag{21}$$

Proof.

$$\begin{aligned}
\int_{-\infty}^{+\infty} (f(x)\delta^{(1)}(x - x_0))\varphi(x)\,\mathrm{d}x = f(x_0)\varphi(x_0) &= \int_{-\infty}^{+\infty} f(x)\varphi(x)\,\mathrm{d}\delta(x - x_0) \\
&= -\int_{-\infty}^{+\infty} f^{(1)}(x)\delta(x - x_0)\varphi(x)\,\mathrm{d}x - \int_{-\infty}^{+\infty} f(x)\delta(x - x_0)\varphi^{(1)}(x)\,\mathrm{d}x \\
&= -\langle f^{(1)}(x)\delta(x - x_0), \varphi(x)\rangle + f(x_0)\langle \delta(x - x_0), -\varphi^{(1)}(x)\rangle \\
&= -f^{(1)}(x_0)\delta(x - x_0) + f(x_0)\delta^{(1)}(x - x_0)
\end{aligned}$$

Similarly, it can be shown that:

$$\begin{aligned}
f(x)\delta^{(n)}(x - x_0) = {} & (-1)^n f^{(n)}(x_0)\delta(x - x_0) + (-1)^{n-1} n f^{(n-1)}(x_0)\delta^{(1)}(x - x_0) + (-1)^{n-2} \\
& \times \frac{n(n-1)}{2!} f^{(n-2)}(x_0)\delta^{(2)}(x - x_0) + \cdots + f(x_0)\delta^{(n)}(x - x_0) \quad \square
\end{aligned} \tag{22}$$

Corollary 2. *The product of Heaviside's function and the nth distributional derivative of the Dirac delta function can be expressed as:*

$$H(x-x_1)\delta^{(n)}(x-x_0) = \begin{cases} H(x_1-x_0)\delta^{(n)}(x-x_0); & x_1 \neq x_0 \\ \textit{not defined}; & x_1 = x_0 \end{cases} \quad n \geqslant 1 \tag{23}$$

Corollary 3. *The nth distributional derivative of the product of a function $f(x)$ and Heaviside's function may be expressed as:*

$$[f(x)H(x-x_0)]^{(n)} = f^{(n)}(x_0)H(x-x_0) + f^{(n-1)}(x_0)\delta(x-x_0) + f^{(n-2)}(x_0)\delta^{(1)}(x-x_0) + \cdots + f(x_0)\delta^{(n-1)}(x-x_0) \tag{24}$$

2.1. Jump discontinuity

Let f be a real valued function on $I=[a,b]$; i.e., f: $[a,b]\,T\nabla$. The function f has a jump discontinuity at $x_0 \in I$ if $f(x_0^+)$ and $f(x_0^-)$ both exist, but f is not continuous at $x = x_0$. The quantity $\Delta f(x_0) = f(x_0^+) - f(x_0^-)$ is called the strength of jump discontinuity at $x = x_0$. There are two possibilities:

(a) $f(x_0^+) \neq f(x_0^-)$
(b) $f(x_0^+) = f(x_0^-) \neq f(x_0)$

In case (a), f has a jump discontinuity, while in case (b), f has a removable discontinuity. A jump discontinuity is also called a simple discontinuity or a discontinuity of the first kind. All other discontinuities for which $f(x_0^+)$ or $f(x_0^-)$ does not exist are called discontinuities of the second kind.

2.2. Functions with jump discontinuities

Consider a classical function $f(x)$ defined in the interval $[0,L]$ with jump discontinuities at $x = x_0$. The function $f(x)$ can be represented by Eq. (14) or Eq. (15). Assume that:

$$f(x_0^+) - f(x_0^-) = \Delta_0^{(0)} \tag{25a}$$

$$\frac{\mathrm{d}f(x_0^+)}{\mathrm{d}x} - \frac{\mathrm{d}f(x_0^-)}{\mathrm{d}x} = \Delta_0^{(1)} \tag{25b}$$

$$\frac{\mathrm{d}^2 f(x_0^+)}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 f(x_0^-)}{\mathrm{d}x^2} = \Delta_0^{(2)} \tag{25c}$$

$$\vdots$$

$$\frac{\mathrm{d}^n f(x_0^+)}{\mathrm{d}x^n} - \frac{\mathrm{d}^n f(x_0^-)}{\mathrm{d}x^n} = \Delta_0^{(n)} \tag{25d}$$

where $\Delta_0^{(k)}$ is the strength of the jump discontinuity in the kth derivative of $f(x)$ at $x = x_0$. Differentiating both sides of Eq. (15) with respect to x in the space of generalized functions yields:

$$\frac{\bar{\mathrm{d}}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}f_1(x)}{\mathrm{d}x} + \left[\frac{\mathrm{d}f_2(x)}{\mathrm{d}x} - \frac{\mathrm{d}f_1(x)}{\mathrm{d}x}\right]H(x-x_0) + [f_2(x) - f_1(x)]\delta(x-x_0) \tag{26}$$

From Eqs. (8) and (26), we have:

$$\frac{\bar{\mathrm{d}}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}f_1(x)}{\mathrm{d}x} + \left[\frac{\mathrm{d}f_2(x)}{\mathrm{d}x} - \frac{\mathrm{d}f_1(x)}{\mathrm{d}x}\right]H(x-x_0) + [f_2(x) - f_1(x)]_{x=x_0}\delta(x-x_0)$$
$$= \frac{\mathrm{d}f_1(x)}{\mathrm{d}x} + \left[\frac{\mathrm{d}f_2(x)}{\mathrm{d}x} - \frac{\mathrm{d}f_1(x)}{\mathrm{d}x}\right]H(x-x_0) + \varDelta_0^{(0)}\delta(x-x_0) \tag{27}$$

Similarly, differentiating both sides of Eq. (27) yields:

$$\frac{\bar{\mathrm{d}}^{(2)}f(x)}{\mathrm{d}x^2} = \frac{\mathrm{d}^2f_1(x)}{\mathrm{d}x^2} + \left[\frac{\mathrm{d}^2f_2(x)}{\mathrm{d}x^2} - \frac{\mathrm{d}^2f_1(x)}{\mathrm{d}x^2}\right]H(x-x_0) + \varDelta_0^{(1)}\delta(x-x_0) + \varDelta_0^{(0)}\delta^{(1)}(x-x_0) \tag{28}$$

Likewise, the kth distributional derivative of $f(x)$ may be written as:

$$\frac{\bar{\mathrm{d}}^{(k)}f(x)}{\mathrm{d}x^k} = \frac{\mathrm{d}^kf_1(x)}{\mathrm{d}x^k} + \left[\frac{\mathrm{d}^kf_2(x)}{\mathrm{d}x^k} - \frac{\mathrm{d}^kf_1(x)}{\mathrm{d}x^k}\right]H(x-x_0) + \varDelta_0^{(k-1)}\delta(x-x_0) + \varDelta_0^{(k-2)}\delta^{(1)}(x-x_0)$$
$$+ \varDelta_0^{(k-3)}\delta^{(2)}(x-x_0) + \cdots + \varDelta_0^{(0)}\delta^{(k-1)}(x-x_0) \tag{29}$$

3. Nonuniform Timoshenko beams with jump discontinuities

In this section, the governing equilibrium equations of a nonuniform Timoshenko beam with jump discontinuities in deflection, slope, and mechanical properties are found in the space of generalized functions. Timoshenko beam theory is based on the following displacement field:

$$u_1(x,y,z) = z\Phi(x) \tag{30a}$$
$$u_2(x,y,z) = 0 \tag{30b}$$
$$u_3(x,y,z) = w^{\mathrm{T}}(x) \tag{30c}$$

where u_1, u_2, and u_3 are displacement components along the beam axis, the beam width, and the beam depth, respectively, and $w^{\mathrm{T}}(x)$ and $\Phi(x)$ are the transverse deflection and rotation about the y-axis, respectively. This theory assumes that all the planar cross-sections perpendicular to the beam axis prior to deformation remain planar after deformation. In this theory a constant shear stress distribution along the beam depth is assumed, which obviously contradicts elasticity boundary conditions for the top and bottom free surfaces of the beam. However, using a shear correction factor, this theory yields good engineering approximations of deflections and forces. Using the principle of virtual work, the governing equilibrium equations of a continuous nonuniform Timoshenko beam may be written as:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(EI(x)\frac{\mathrm{d}\Phi(x)}{\mathrm{d}x}\right) - GA_{\mathrm{s}}(x)\left(\Phi(x) + \frac{\mathrm{d}w^{\mathrm{T}}(x)}{\mathrm{d}x}\right) = 0 \tag{31a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[GA_{\mathrm{s}}(x)\left(\Phi(x) + \frac{\mathrm{d}w^{\mathrm{T}}(x)}{\mathrm{d}x}\right)\right] + q(x) = 0 \tag{31b}$$

where $EI(x)$ and $GA_{\mathrm{s}}(x)$ are the respective flexural and shear stiffnesses and $q(x)$ is a distributed lateral force. Performing the differentiations in Eqs. (31a) and (31b) yields:

$$\frac{\mathrm{d}^2\Phi(x)}{\mathrm{d}x^2} + \frac{EI'(x)}{EI(x)}\frac{\mathrm{d}\Phi(x)}{\mathrm{d}x} - \frac{GA'_{\mathrm{s}}(x)}{EI(x)}\left(\Phi(x) + \frac{\mathrm{d}w^{\mathrm{T}}(x)}{\mathrm{d}x}\right) = 0 \tag{32a}$$

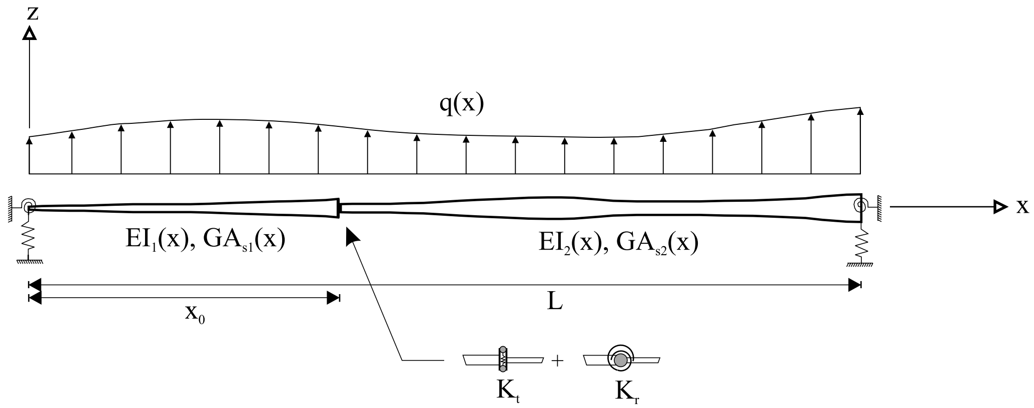

Fig. 2. A nonuniform beam with jump discontinuities in slope, deflection, and mechanical properties.

$$\frac{d^2 w^T(x)}{dx^2} + \frac{d\Phi(x)}{dx} + \frac{GA_s'(x)}{GA_s(x)}\left(\Phi(x) + \frac{dw^T(x)}{dx}\right) + \frac{q(x)}{GA_s(x)} = 0 \tag{32b}$$

where $(\cdot)'$ denotes classical differentiation with respect to x.

Now consider the beam shown in Fig. 2. This beam has one point of discontinuity. At this point both deflection and slope have jump discontinuities, and flexural and shear stiffnesses change abruptly. The results obtained for this beam can easily be generalized for a beam with n points of discontinuity. This beam is composed of two beam segments 1 and 2 in the intervals $0 \leqslant x < x_0$ and $x_0 \leqslant x < L$ with respective deflections and rotations $w_1^T(x)$, $\Phi_1(x)$, $w_2^T(x)$, and $\Phi_2(x)$. For each continuous beam segment, Eqs. (32a) and (32b) are the governing equilibrium equations. Hence for $0 \leqslant x < x_0$:

$$\frac{d^2 \Phi_1(x)}{dx^2} + \frac{EI_1'(x)}{EI_1(x)}\frac{d\Phi_1(x)}{dx} - \frac{GA_{s1}'(x)}{EI_1(x)}\left(\Phi_1(x) + \frac{dw_1^T(x)}{dx}\right) = 0 \tag{33a}$$

$$\frac{d^2 w_1^T(x)}{dx^2} + \frac{d\Phi_1(x)}{dx} + \frac{GA_{s1}'(x)}{GA_{s1}(x)}\left(\Phi_1(x) + \frac{dw_1^T(x)}{dx}\right) + \frac{q(x)}{GA_{s1}(x)} = 0 \tag{33b}$$

and for $x_0 \leqslant x < L$:

$$\frac{d^2 \Phi_2(x)}{dx^2} + \frac{EI_2'(x)}{EI_2(x)}\frac{d\Phi_2(x)}{dx} - \frac{GA_{s2}'(x)}{EI_2(x)}\left(\Phi_2(x) + \frac{dw_2^T(x)}{dx}\right) = 0 \tag{34a}$$

$$\frac{d^2 w_2^T(x)}{dx^2} + \frac{d\Phi_2(x)}{dx} + \frac{GA_{s2}'(x)}{GA_{s2}(x)}\left(\Phi_2(x) + \frac{dw_2^T(x)}{dx}\right) + \frac{q(x)}{GA_{s2}(x)} = 0 \tag{34b}$$

Now let:

$$\frac{EI_i'(x)}{EI_i(x)} = \alpha_i(x), \quad \frac{GA_{si}'(x)}{EI_i(x)} = \beta_i(x), \quad \frac{GA_{si}'(x)}{GA_{si}(x)} = \gamma_i(x); \qquad i = 1, 2 \tag{35}$$

Substituting Eq. (35) into Eqs. (33a), (33b), (34a) and (34b) yields:

$$\frac{d^2 \Phi_1(x)}{dx^2} + \alpha_1(x)\frac{d\Phi_1(x)}{dx} - \beta_1(x)\left(\Phi_1(x) + \frac{dw_1^T(x)}{dx}\right) = 0 \tag{36a}$$

$$\frac{d^2 w_1^T(x)}{dx^2} + \frac{d\Phi_1(x)}{dx} + \gamma_1(x)\left(\Phi_1(x) + \frac{dw_1^T(x)}{dx}\right) + \frac{q(x)}{GA_{s1}(x)} = 0 \tag{36b}$$

and

$$\frac{d^2\Phi_2(x)}{dx^2} + \alpha_2(x)\frac{d\Phi_2(x)}{dx} - \beta_2(x)\left(\Phi_2(x) + \frac{dw_2^T(x)}{dx}\right) = 0 \tag{37a}$$

$$\frac{d^2 w_2^T(x)}{dx^2} + \frac{d\Phi_2(x)}{dx} + \gamma_2(x)\left(\Phi_2(x) + \frac{dw_2^T(x)}{dx}\right) + \frac{q(x)}{GA_{s2}(x)} = 0 \tag{37b}$$

Denote the deflection and rotation of the beam by $w^T(x)$ and $\Phi(x)$, respectively. These functions have jump discontinuities at $x = x_0$:

$$w^T(x) = \begin{cases} w_1^T(x); & 0 \leqslant x < x_0 \\ w_2^T(x); & x_0 < x \leqslant L \end{cases} \quad \text{and} \quad \Phi(x) = \begin{cases} \Phi_1(x); & 0 \leqslant x < x_0 \\ \Phi_2(x); & x_0 < x \leqslant L \end{cases} \tag{38}$$

Proceeding as we did with Eq. (14), we can write:

$$w^T(x) = w_1^T(x) + \left[w_2^T(x) - w_1^T(x)\right]H(x - x_0) \tag{39a}$$

$$\Phi(x) = \Phi_1(x) + [\Phi_2(x) - \Phi_1(x)]H(x - x_0) \tag{39b}$$

Now let:

$$w^T(x_0^+) - w^T(x_0^-) = w_2^T(x_0) - w_1^T(x_0) = \Delta^T \tag{40a}$$

$$\Phi(x_0^+) - \Phi(x_0^-) = \Phi_2(x_0) - \Phi_1(x_0) = \Theta^T \tag{40b}$$

where Δ^T and Θ^T are the strengths of jump discontinuities in deflection and slope, respectively. We also know that:

$$M_1(x_0) = EI_1(x_0)\frac{d\Phi_1(x_0)}{dx}, \quad M_2(x_0) = EI_2(x_0)\frac{d\Phi_2(x_0)}{dx} \tag{41a}$$

$$V_1(x_0) = GA_{s1}(x_0)\left(\Phi_1(x_0) + \frac{dw_1^T(x_0)}{dx}\right), \quad V_2(x_0) = GA_{s2}(x_0)\left(\Phi_2(x_0) + \frac{dw_2^T(x_0)}{dx}\right) \tag{41b}$$

where the M_is and V_is are bending moments and shear forces, respectively. From Eqs. (40a), (40b), (41a) and (41b), we obtain:

$$\left[\frac{dw_2^T(x)}{dx} - \frac{dw_1^T(x)}{dx}\right]_{x=x_0} = K_t\Delta^T\left(\frac{1}{GA_{s2}(x_0)} - \frac{1}{GA_{s1}(x_0)}\right) - \Theta^T = a_0\Delta^T - \Theta^T \tag{42a}$$

$$\left[\frac{d\Phi_2(x)}{dx} - \frac{d\Phi_1(x)}{dx}\right]_{x=x_0} = K_r\Theta^T\left(\frac{1}{EI_2(x_0)} - \frac{1}{EI_1(x_0)}\right) = b_0\Theta^T \tag{42b}$$

Using Eqs. (29), (40a) and (40b), we obtain:

$$\frac{\bar{d}w^T(x)}{dx} = \frac{dw_1^T(x)}{dx} + \left[\frac{dw_2^T(x)}{dx} - \frac{dw_1^T(x)}{dx}\right]H(x - x_0) + \Delta^T\delta(x - x_0) \tag{43a}$$

$$\frac{\bar{d}^2 w^T(x)}{dx^2} = \frac{d^2 w_1^T(x)}{dx_2} + \left[\frac{d^2 w_2^T(x)}{dx^2} - \frac{d^2 w_1^T(x)}{dx^2}\right]H(x - x_0) + (a_0\Delta^T - \Theta^T)\delta(x - x_0) + \Delta^T\delta^{(1)}(x - x_0) \tag{43b}$$

$$\frac{\bar{\mathrm{d}}\Phi(x)}{\mathrm{d}x}=\frac{\mathrm{d}\Phi_1(x)}{\mathrm{d}x}+\left[\frac{\mathrm{d}\Phi_2(x)}{\mathrm{d}x}-\frac{\mathrm{d}\Phi_1(x)}{\mathrm{d}x}\right]H(x-x_0)+\Theta^{\mathrm{T}}\delta(x-x_0) \tag{43c}$$

$$\frac{\bar{\mathrm{d}}^2\Phi(x)}{\mathrm{d}x^2}=\frac{\mathrm{d}^2\Phi_1(x)}{\mathrm{d}x_2}+\left[\frac{\mathrm{d}^2\Phi_2(x)}{\mathrm{d}x^2}-\frac{\mathrm{d}^2\Phi_1(x)}{\mathrm{d}x^2}\right]H(x-x_0)+b_0\Theta^{\mathrm{T}}\delta(x-x_0)+\Delta^{\mathrm{T}}\delta^{(1)}(x-x_0) \tag{43d}$$

Denoting the left-hand sides of Eqs. (36a) and (37a) by $\mathfrak{I}$ and $\mathfrak{R}$, respectively, we can write:

$$\mathfrak{I}+(\mathfrak{R}-\mathfrak{I})H(x-x_0)=0. \tag{44}$$

Substituting for $\mathfrak{I}$ and $\mathfrak{R}$ from Eqs. (36a) and (37a) into Eq. (44) yields:

$$\begin{aligned}&\frac{\mathrm{d}^2\Phi_1(x)}{\mathrm{d}x^2}+\alpha_1(x)\frac{\mathrm{d}\Phi_1(x)}{\mathrm{d}x}-\beta_1(x)\left(\Phi_1(x)+\frac{\mathrm{d}w_1^{\mathrm{T}}(x)}{\mathrm{d}x}\right)+\left[\frac{\mathrm{d}^2\Phi_2(x)}{\mathrm{d}x^2}-\frac{\mathrm{d}^2\Phi_1(x)}{\mathrm{d}x^2}+\alpha_2(x)\frac{\mathrm{d}\Phi_2(x)}{\mathrm{d}x}\right.\\&\left.\quad-\alpha_1(x)\frac{\mathrm{d}\Phi_1(x)}{\mathrm{d}x}-\beta_2(x)\left(\Phi_1(x)+\frac{\mathrm{d}w_1^{\mathrm{T}}(x)}{\mathrm{d}x}\right)+\beta_1(x)\left(\Phi_1(x)+\frac{\mathrm{d}w_1^{\mathrm{T}}(x)}{\mathrm{d}x}\right)\right]H(x-x_0)=0\end{aligned} \tag{45}$$

or

$$\begin{aligned}&\frac{\mathrm{d}^2\Phi_1(x)}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}^2\Phi_2(x)}{\mathrm{d}x^2}-\frac{\mathrm{d}^2\Phi_1(x)}{\mathrm{d}x^2}\right)H(x-x_0)+\alpha_1(x)\left[\frac{\mathrm{d}\Phi_1(x)}{\mathrm{d}x}+\left(\frac{\mathrm{d}\Phi_1(x)}{\mathrm{d}x}-\frac{\mathrm{d}\Phi_1(x)}{\mathrm{d}x}\right)H(x-x_0)\right]\\&\quad+(\alpha_2(x)-\alpha_1(x))\frac{\mathrm{d}\Phi_2(x)}{\mathrm{d}x}H(x-x_0)-\beta_1(x)[\Phi_1(x)+(\Phi_2(x)-\Phi_1(x))H(x-x_0)]\\&\quad+(\beta_1(x)-\beta_2(x))\frac{\mathrm{d}w_2^{\mathrm{T}}(x)}{\mathrm{d}x}H(x-x_0)=0\end{aligned} \tag{46}$$

From Eqs. (46), (43c), and (43d), we obtain:

$$\begin{aligned}&\frac{\bar{\mathrm{d}}^2\Phi(x)}{\mathrm{d}x^2}+\alpha_1(x)\frac{\bar{\mathrm{d}}\Phi(x)}{\mathrm{d}x}+(\alpha_2(x)-\alpha_1(x))\frac{\mathrm{d}\Phi_2(x)}{\mathrm{d}x}H(x-x_0)-\beta_1(x)\frac{\bar{\mathrm{d}}w^{\mathrm{T}}(x)}{\mathrm{d}x}\\&\quad+(\beta_1(x)-\beta_2(x))\frac{\mathrm{d}w_2^{\mathrm{T}}(x)}{\mathrm{d}x}H(x-x_0)=\left[(b_0+\alpha_1(x))\Theta^{\mathrm{T}}-\beta_1(x)\Delta^{\mathrm{T}}\right]\delta(x-x_0)+\Theta^{\mathrm{T}}\delta^{(1)}(x-x_0)\end{aligned} \tag{47}$$

Multiplying both sides of Eq. (43a) and considering Eq. (19b) yields:

$$\begin{aligned}\frac{\bar{\mathrm{d}}w^{\mathrm{T}}(x)}{\mathrm{d}x}H(x-x_0)&=\frac{\mathrm{d}w_2^{\mathrm{T}}(x)}{\mathrm{d}x}H(x-x_0)+\Delta^{\mathrm{T}}H(x-x_0)\delta(x-x_0)\\&=\frac{\mathrm{d}w_2^{\mathrm{T}}(x)}{\mathrm{d}x}H(x-x_0)+\frac{\Delta^{\mathrm{T}}}{2}\delta(x-x_0)\end{aligned} \tag{48}$$

Thus:

$$\frac{\mathrm{d}w_2^{\mathrm{T}}(x)}{\mathrm{d}x}H(x-x_0)=\frac{\bar{\mathrm{d}}w^{\mathrm{T}}(x)}{\mathrm{d}x}H(x-x_0)-\frac{\Delta^{\mathrm{T}}}{2}\delta(x-x_0) \tag{49}$$

Substituting Eq. (49) into Eq. (47) yields:

$$\begin{aligned}&\frac{\bar{\mathrm{d}}^2\Phi(x)}{\mathrm{d}x^2}+[\alpha_1(x)+(\alpha_2(x)-\alpha_1(x))H(x-x_0)]\frac{\bar{\mathrm{d}}\Phi(x)}{\mathrm{d}x}-[\alpha_1(x)+(\alpha_2(x)-\alpha_1(x))H(x-x_0)]\\&\quad\times\left(\Phi(x)+\frac{\bar{\mathrm{d}}w^{\mathrm{T}}(x)}{\mathrm{d}x}\right)=\left[\left(b_0+\frac{\alpha_1(x)+\alpha_2(x)}{2}\right)\Theta^{\mathrm{T}}-\frac{\beta_1(x)+\beta_2(x)}{2}\Delta^{\mathrm{T}}\right]\delta(x-x_0)+\Theta^{\mathrm{T}}\delta^{(1)}(x-x_0)\end{aligned} \tag{50}$$

Similarly from Eqs. (36b) and (37b), we obtain:

$$\frac{\bar{\mathrm{d}}^2 w^{\mathrm{T}}(x)}{\mathrm{d}x^2} + [\gamma_1(x) + (\gamma_2(x) - \gamma_1(x))H(x - x_0)]\frac{\bar{\mathrm{d}} w^{\mathrm{T}}(x)}{\mathrm{d}x} + \frac{\bar{\mathrm{d}}\Phi(x)}{\mathrm{d}x} + [\gamma_1(x) + (\gamma_2(x) - \gamma_1(x))H(x - x_0)]\Phi(x)$$
$$= \left(a_0 + \frac{\gamma_1(x) + \gamma_2(x)}{2}\right)\Delta^{\mathrm{T}}\delta(x - x_0) + \Delta^{\mathrm{T}}\delta^{(1)}(x - x_0) \tag{51}$$

Eqs. (50) and (51) are the equilibrium differential equations of a nonuniform Timoshenko beam with one point of jump discontinuity in the space of generalized functions. As may be seen, the governing differential equilibrium equations can always be expressed in terms of a single displacement function $w(x)$ and a single rotation function $\Phi(x)$.

In Section 4 we study the bending of nonuniform Euler–Bernoulli beams with jump discontinuities. There we demonstrate that, unlike what we found for Timoshenko beams, the governing differential equilibrium equation of the corresponding Euler–Bernoulli beams cannot always be written in terms of a single deflection function $w(x)$.

4. Nonuniform Euler–Bernoulli beams with jump discontinuities

In this section, bending of nonuniform Euler–Bernoulli beams with jump discontinuities is investigated. Euler–Bernoulli beam theory, which neglects shear deformation, is the simplest theory of beams. This theory assumes that any planar cross-section perpendicular to the beam axis prior to deformation remains planar (no warping) and perpendicular to the beam axis after deformation. For this theory the following displacement field is assumed:

$$u_1(x,y,z) = -z\frac{\mathrm{d}w(x)}{\mathrm{d}x} \tag{52a}$$

$$u_2(x,y,z) = 0 \tag{52b}$$

$$u_3(x,y,z) = w(x) \tag{52c}$$

where u_1, u_2, u_3, and w have the same definitions as those presented for Timoshenko beams in the previous section. The governing differential equilibrium equation of a continuous nonuniform Euler–Bernoulli beam may be written as:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[EI(x)\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}\right] = q(x) \tag{53}$$

where $EI(x)$ is the flexural stiffness and $q(x)$ is a distributed force. Performing the differentiations in Eq. (53), we obtain:

$$\frac{\mathrm{d}^4 w(x)}{\mathrm{d}x^4} + \frac{2EI'(x)}{EI(x)}\frac{\mathrm{d}^3 w(x)}{\mathrm{d}x^3} + \frac{EI''(x)}{EI(x)}\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} = \frac{q(x)}{EI(x)} \tag{54}$$

where $(\cdot)'$ and $(\cdot)''$ denote classic differentiations with respect to x. Now let:

$$\frac{2EI'(x)}{EI(x)} = J(x) \quad \text{and} \quad \frac{EI''(x)}{EI(x)} = K(x) \tag{55}$$

Substituting Eq. (55) into Eq. (54) yields:

$$\frac{\mathrm{d}^4 w(x)}{\mathrm{d}x^4} + J(x)\frac{\mathrm{d}^3 w(x)}{\mathrm{d}x^3} + K(x)\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} = \frac{q(x)}{EI(x)} \tag{56}$$

Obviously Eq. (56) is not the governing equilibrium equation of a discontinuous Euler–Bernoulli beam; it must be modified somewhat before it is relevant to discontinuous beams.

Consider the beam shown in Fig.2. Again, for the sake of simplicity, only one point of jump discontinuity is considered. This beam is composed of two beam segments with respective deflections $w_1(x)$ and $w_2(x)$. Each beam segment is continuous and hence Eq. (56) is the governing equilibrium equation. Hence:

$$\frac{\mathrm{d}^4 w_1(x)}{\mathrm{d}x^4} + J_1(x)\frac{\mathrm{d}^3 w_1(x)}{\mathrm{d}x^3} + K_1(x)\frac{\mathrm{d}^2 w_1(x)}{\mathrm{d}x^2} = \frac{q(x)}{EI_1(x)}; \quad 0 \leqslant x < x_0 \tag{57a}$$

$$\frac{\mathrm{d}^4 w_2(x)}{\mathrm{d}x^4} + J_2(x)\frac{\mathrm{d}^3 w_2(x)}{\mathrm{d}x^3} + K_2(x)\frac{\mathrm{d}^2 w_2(x)}{\mathrm{d}x^2} = \frac{q(x)}{EI_2(x)}; \quad x_0 < x \leqslant L \tag{57b}$$

The displacement function of the beam is a piecewise continuous function that has the following compact representation, which is similar to the representation of Timoshenko beams shown earlier:

$$w(x) = w_1(x) + [w_2(x) - w_1(x)]H(x - x_0) \tag{58}$$

For this discontinuous beam we have:

$$w(x_0^+) - w(x_0^-) = w_2(x_0) - w_1(x_0) = \varDelta \tag{59}$$

Also we know that:

$$M_1(x_0) = EI_1(x_0)\frac{\mathrm{d}^2 w_1(x_0)}{\mathrm{d}x^2}, \quad M_2(x_0) = EI_2(x_0)\frac{\mathrm{d}^2 w_2(x_0)}{\mathrm{d}x^2} \tag{60a}$$

$$V_1(x_0) = EI_1(x_0)\frac{\mathrm{d}^3 w_1(x_0)}{\mathrm{d}x^3}, \quad V_2(x_0) = EI_2(x_0)\frac{\mathrm{d}^3 w_2(x_0)}{\mathrm{d}x^3} \tag{60b}$$

Thus:

$$\left[\frac{\mathrm{d}^2 w_2(x)}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 w_1(x)}{\mathrm{d}x^2}\right]_{x=x_0} = K_\mathrm{r}\varTheta\left(\frac{1}{EI_2(x_0)} - \frac{1}{EI_1(x_0)}\right) = K_\mathrm{r} a_0 \varTheta \tag{61a}$$

$$\left[\frac{\mathrm{d}^3 w_2(x)}{\mathrm{d}x^3} - \frac{\mathrm{d}^3 w_1(x)}{\mathrm{d}x^3}\right]_{x=x_0} = K_\mathrm{t}\varDelta\left(\frac{1}{EI_2(x_0)} - \frac{1}{EI_1(x_0)}\right) = K_\mathrm{t} a_0 \varDelta \tag{61b}$$

Differentiating both sides of Eq. (58) and considering Eq. (29) yields:

$$\frac{\bar{\mathrm{d}} w(x)}{\mathrm{d}x} = \frac{\mathrm{d}w_1(x)}{\mathrm{d}x} + \left[\frac{\mathrm{d}w_2(x)}{\mathrm{d}x} - \frac{\mathrm{d}w_1(x)}{\mathrm{d}x}\right]H(x - x_0) + \varDelta\delta(x - x_0) \tag{62a}$$

$$\frac{\bar{\mathrm{d}}^2 w(x)}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 w_1(x)}{\mathrm{d}x^3} + \left[\frac{\mathrm{d}^2 w_2(x)}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 w_1(x)}{\mathrm{d}x^2}\right]H(x - x_0) + \varTheta\delta(x - x_0) + \varDelta\delta^{(1)}(x - x_0) \tag{62b}$$

$$\frac{\bar{\mathrm{d}}^3 w(x)}{\mathrm{d}x^3} = \frac{\mathrm{d}^3 w_1(x)}{\mathrm{d}x^3} + \left[\frac{\mathrm{d}^3 w_2(x)}{\mathrm{d}x^3} - \frac{\mathrm{d}^3 w_1(x)}{\mathrm{d}x^3}\right]H(x - x_0) + K_\mathrm{r} a_0 \varTheta\delta(x - x_0) + \varTheta\delta^{(1)}(x - x_0) + \varDelta\delta^{(2)}(x - x_0) \tag{62c}$$

$$\frac{\bar{d}^4 w(x)}{dx^4} = \frac{d^4 w_1(x)}{dx^4} + \left[\frac{d^4 w_2(x)}{dx^4} - \frac{d^4 w_1(x)}{dx^4}\right] H(x-x_0) + K_t a_0 \Delta \delta(x-x_0) + K_r a_0 \Theta \delta^{(1)}(x-x_0) + \Theta \delta^{(2)}(x-x_0) + \Delta \delta^{(3)}(x-x_0) \quad (62d)$$

Denoting the left-hand sides of Eqs. (57a) and (57b) by $\mathfrak{I}$ and $\mathfrak{R}$, respectively, we can write:

$$\mathfrak{I} + (\mathfrak{R} - \mathfrak{I}) H(x-x_0) = 0 \quad (63)$$

Substituting for $\mathfrak{I}$ and $\mathfrak{R}$ from Eqs. (57a) and (57b) into Eq. (63) yields:

$$\frac{d^4 w_1(x)}{dx^4} + \left(\frac{d^4 w_2(x)}{dx^4} - \frac{d^4 w_1(x)}{dx^4}\right) H(x-x_0) + J_1(x)\left[\frac{d^3 w_1(x)}{dx^3} + \left(\frac{d^3 w_2(x)}{dx^3} - \frac{d^3 w_1(x)}{dx^3}\right) H(x-x_0)\right] + K_1(x)\left[\frac{d^2 w_1(x)}{dx^2} + \left(\frac{d^2 w_2(x)}{dx^2} - \frac{d^2 w_1(x)}{dx^2}\right) H(x-x_0)\right] + (J_2(x) - J_1(x))\frac{d^3 w_2(x)}{dx^3} H(x-x_0) + (K_2(x) - K_1(x))\frac{d^2 w_2(x)}{dx^2} H(x-x_0) = q(x)\left[\frac{1}{EI_1(x)} + \left(\frac{1}{EI_2(x)} - \frac{1}{EI_1(x)}\right) H(x-x_0)\right] \quad (64)$$

Multiplying both sides of Eqs. (62b) and (62c) by $H(x-x_0)$ yields:

$$\frac{\bar{d}^2 w(x)}{dx} H(x-x_0) = \frac{d^2 w_2(x)}{dx} H(x-x_0) + \Theta \delta(x-x_0) H(x-x_0) + \Delta \delta^{(1)}(x-x_0) H(x-x_0) \quad (65a)$$

$$\frac{\bar{d}^3 w(x)}{dx} H(x-x_0) = \frac{d^3 w_2(x)}{dx} H(x-x_0) + K_r a_0 \Theta \delta(x-x_0) H(x-x_0) + \Theta \delta^{(1)}(x-x_0) H(x-x_0) + \Delta \delta^{(2)}(x-x_0) H(x-x_0) \quad (65b)$$

From Eqs. (64), (65a), (65b) and considering Eqs. (19b) and (23), it can be concluded that $w_2(x)$ can be eliminated from the governing differential equilibrium equation if and only if:

$$\Delta = 0 \quad \text{or} \quad K_1(x) = K_2(x) \quad (66a)$$

and

$$\Theta = 0 \quad \text{or} \quad J_1(x) = J_2(x) \quad (66b)$$

In other words, the governing equilibrium equation of a nonuniform Euler–Bernoulli beam with jump discontinuities can be expressed in terms of a single deflection function $w(x)$ in the space of generalized functions if and only if both of the following conditions hold:

$$\Delta = 0 \quad \text{or} \quad \frac{EI_1''(x)}{EI_1(x)} = \frac{EI_2''(x)}{EI_2(x)} \quad (67a)$$

$$\Theta = 0 \quad \text{or} \quad \frac{EI_1'(x)}{EI_1(x)} = \frac{EI_2'(x)}{EI_2(x)} \quad (67b)$$

Apparently, conditions (67a) and (67b) are very restrictive, and for most nonuniform Euler–Bernoulli beams with jump discontinuities it is not possible to express the governing differential equilibrium equation in terms of a single function; two deflection functions or more appear in the governing equations. Clearly, uniform Euler–Bernoulli beams satisfy both Eqs. (67a) and (67b) and hence the governing equilibrium equation can always be written in terms of a single displacement function $w(x)$, as was shown by Yavari et al. (2000).

Now a more general result can be reached. Consider a mechanical system H whose governing differential equation has the following form:

$$\frac{d^n f(x)}{dx^n} + b_{n-1}(x)\frac{d^{n-1}f(x)}{dx^{n-1}} + \cdots + b_2(x)\frac{d^2 f(x)}{dx^2} + b_1(x)\frac{df(x)}{dx} + b_0(x)f(x) = q(x) \tag{68}$$

for $x \in [a,b]$. It is assumed that the response of the system is represented by a function $f(x)$ when there are no discontinuities. Now suppose that this system has a discontinuity point at $x = x_0$; the system is composed of two continuous subsystems. Generalization to the case of n discontinuity points is straightforward. Denoting the respective responses of subsystems by $f_1(x)$ and $f_2(x)$, Eq. (68) is the governing differential equation for each of them, i.e.:

$$\frac{d^n f_1(x)}{dx^n} + {}^1b_{n-1}(x)\frac{d^{n-1}f_1(x)}{dx^{n-1}} + \cdots + {}^1b_2(x)\frac{d^2 f_1(x)}{dx^2} + {}^1b_1(x)\frac{df_1(x)}{dx} + {}^1b_0(x)f_1(x) = q(x) \tag{69a}$$

for $x \in [a,x_0]$, and

$$\frac{d^n f_2(x)}{dx^n} + {}^2b_{n-1}(x)\frac{d^{n-1}f_2(x)}{dx^{n-1}} + \cdots + {}^2b_2(x)\frac{d^2 f_2(x)}{dx^2} + {}^2b_1(x)\frac{df_2(x)}{dx} + {}^2b_0(x)f_2(x) = q(x) \tag{69b}$$

for $x \in [x_0,b]$. For this system we have:

$$f(x_0^+) - f(x_0^-) = f_2(x_0) - f_1(x_0) = \Delta_0^{(0)} \tag{70a}$$

$$f^{(1)}(x_0^+) - f^{(1)}(x_0^-) = f_2^{(1)}(x_0) - f_1^{(1)}(x_0) = \Delta_0^{(1)} \tag{70b}$$

$$f^{(2)}(x_0^+) - f^{(2)}(x_0^-) = f_2^{(2)}(x_0) - f_1^{(2)}(x_0) = \Delta_0^{(2)} \tag{70c}$$

$$\vdots$$

$$f^{(n-1)}(x_0^+) - f^{(n-1)}(x_0^-) = f_2^{(n-1)}(x_0) - f_1^{(n-1)}(x_0) = \Delta_0^{(n-1)} \tag{70d}$$

Theorem 2. *The governing differential equation of system H can be written in the space of generalized functions in terms of $f(x)$ if and only if:*

$${}^1b_k(x) = {}^2b_k(x) \quad \text{or} \quad \Delta_0^{(k-2)} = \Delta_0^{(k-3)} = \cdots = \Delta_0^{(0)} = 0; \quad k = n-1, n-2, \ldots, 2 \tag{71}$$

Proof. Consider the terms ${}^1b_k(x)d^k f_1(x)/dx^k$ and ${}^2b_k(x)d^k f_2(x)/dx^k$. Again following a procedure similar to that used with the Timoshenko and Euler–Bernoulli beams, consider the following combination of these terms:

$$\begin{aligned} &{}^1b_k(x)\frac{d^k f_1(x)}{dx^k} + \left[{}^2b_k(x)\frac{d^k f_2(x)}{dx^k} - {}^1b_k(x)\frac{d^k f_1(x)}{dx^k}\right]H(x-x_0) \\ &\quad = {}^1b_k(x)\left[\frac{d^k f_1(x)}{dx^k} + \left(\frac{d^k f_2(x)}{dx^k} - \frac{d^k f_1(x)}{dx^k}\right)H(x-x_0)\right] + \left({}^2b_k(x) - {}^1b_k(x)\right)\frac{d^k f_2(x)}{dx^k} \end{aligned} \tag{72}$$

The problem is the last term. Multiplying both sides of Eq. (29) by $H(x-x_0)$ yields:

$$\begin{aligned} \frac{\bar{d}^k f(x)}{dx^k}H(x-x_0) = {} & \frac{d^k f_2(x)}{dx^k}H(x-x_0) + \Delta_0^{(k-1)}\delta(x-x_0)H(x-x_0) + \Delta_0^{(k-2)}\delta^{(1)}(x-x_0)H(x-x_0) \\ & + \Delta_0^{(k-3)}\delta^{(2)}(x-x_0)H(x-x_0) + \cdots + \Delta_0^{(0)}\delta^{(k-1)}(x-x_0)H(x-x_0) \end{aligned} \tag{73}$$

Hence, conditions (71) must be satisfied. □

Example 1. Consider an Euler–Bernoulli beam column with jump discontinuities in the deflection and slope and an abrupt change in flexural rigidity with an axial load at the point of discontinuity. Also assume that for each beam-column segment the flexural stiffness is constant. For this mechanical system the governing equilibrium equations of beam-column segments may be written as:

$$\frac{\mathrm{d}^4 w_1(x)}{\mathrm{d}x^4} + \frac{P_1}{EI_1}\frac{\mathrm{d}^2 w_1(x)}{\mathrm{d}x^2} = \frac{q(x)}{EI_1}; \quad 0 \leqslant x < x_0 \tag{74a}$$

$$\frac{\mathrm{d}^4 w_2(x)}{\mathrm{d}x^4} + \frac{P_2}{EI_2}\frac{\mathrm{d}^2 w_2(x)}{\mathrm{d}x^2} = \frac{q(x)}{EI_2}; \quad x_0 \leqslant x < L \tag{74b}$$

For this mechanical system $n = 4$, ${}^1b_3(x) = {}^2b_3(x) = 0$, and ${}^1b_2(x) = P_1/EI_1$, and ${}^2b_2(x) = P_2/EI_2$.

According to the above theorem the governing equilibrium of the beam column can be written in terms of a single deflection function $w(x)$ if and only if:

$$\frac{P_1}{EI_1} = \frac{P_2}{EI_2} \quad \text{or} \quad \Delta_0^{(0)} = 0 \tag{75}$$

which is exactly what was obtained by Yavari and Sarkani (2001).

5. Conclusions

In this article the bending of nonuniform Euler–Bernoulli and Timoshenko beams with various discontinuities is studied. Bending equations for a nonuniform Timoshenko beam with jump discontinuities in deflection and slope, and with abrupt changes in flexural and shear rigidities, are found in the space of generalized functions. It is shown that the governing differential equilibrium equations can always be expressed in terms of a single deflection function and a single rotation function.

Because the product of two generalized functions is not defined in general, for nonuniform Euler–Bernoulli beams with jump discontinuities the governing equilibrium equation cannot always be written in terms of a single deflection function. Conditions under which the governing equilibrium equation can be simplified in terms of only one deflection function are found. Then the same problem is investigated for a more general mechanical system and a theorem is derived. This theorem justifies the results obtained by Yavari and Sarkani (2001) for Euler–Bernoulli beam columns with jump discontinuities.

References

Al-Gahtani, H.J., Khan, M.S., 1998. Exact analysis of nonprismatic beams. ASCE Journal of Engineering Mechanics 124 (11), 1290–1293.

Dirac, P.A.M., 1930. In: The Principles of Quantum Mechanics. Clarendon Press, Oxford, pp. 63–66.

Fertis, D.G., Keene, M.E., 1990. Elastic and inelastic analysis of nonprismatic members. ASCE Journal of Structural Engineering 116 (2), 475–489.

Hetényi, M., 1937. Deflections of beams of varying cross-section. Journal of Applied Mechanics 4 (2), 49–52.

Jones, D.S., 1982. The Theory of Generalized Functions, second ed. Cambridge University Press, London.

Lee, S.Y., Ke, H.Y., Kuo, Y.H., 1990. Exact static deflection of a non-uniform Bernoulli–Euler beam with general elastic end restraints. Computers and Structures 36 (1), 91–97.

Lighthill, M.J., 1958. An Introduction to Fourier Analysis and Generalized Functions. Cambridge University Press, London.

Romano, F., Zingone, G., 1992. Deflection of beams with varying rectangular cross-section. ASCE Journal of Engineering Mechanics 118 (10), 2128–2134.

Schwarz, L., 1966.. Théorie des Distributions, Herman, Paris.

Sobolev, S., 1938. Sur une théorème de l'anayse fonctionelle. Matematiceski Sbornik 4, 471–496.

Yavari, A., Sarkani, S., Moyer Jr., E.T., 2000. On applications of generalized functions to beam bending problems. International Journal of Solids and Structures 37 (40), 5675–5705.

Yavari, A., Sarkani, S., 2001. On applications of generalized functions to the analysis of Euler–Bernoulli beam-columns with jump discontinuities. International Journal of Mechanical Sciences 43, 1543–1562.

Yavari, A., Sarkani, S., Reddy, J.N., 2001. Generalized solutions for beams with jump discontinuities on elastic foundations. Archive of Applied Mechanics, in press.

Zemanian, A.H., 1965. Distribution Theory and Transform Analysis. McGraw-Hill, New York.